CONTROLLABILITY OF SECOND ORDER DELAY INTEGRODIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS

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Abstract

In this paper, we shall establish sufficient conditions for the controllability of second order delay integrodifferential inclusions in Banach spaces, with nonlocal conditions. By using suitable fixed point theorems we study the case when the multivalued map has convex as well nonconvex values.

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1 Introduction

In this paper, we shall establish sufficient conditions for the controllability of second order delay integrodifferential inclusions in Banach spaces, with nonlocal initial conditions. More precisely we consider the following semilinear system of the form

$$y'' - Ay \in \int_0^t K(t,s)F(s,y(\sigma(s)))ds + (Bu)(t), \quad t \in J := [0,b], \tag{1}$$

$$y(0) + f(y) = y_0, \; y'(0) = \eta, \tag{2}$$

where $F : J \times E \longrightarrow \mathcal{P}(E)$ is a multivalued map, $\sigma : J \to J$ is a continuous function such that $\sigma(t) \leq t, \forall t \in J$, $K : D \longrightarrow \mathbb{R}$, $D = \{(t,s) \in J \times J : t \geq s\}$, $f : C(J,E) \longrightarrow E$ is a continuous given function, A is a linear infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in a separable Banach space $E = (E, |\cdot|), y_0, \eta \in E$. Also the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. Finally B is a bounded linear operator from U to E.

The pioneering work on nonlocal evolution Cauchy problems is due to Byszewski. As pointed out by Byszewski [12], [11] the study of initial value problems with nonlocal conditions is of significance since they have applications in problems in physics and other areas of applied mathematics. In fact, more authors have paid attention to the research of initial value problems with nonlocal conditions, in the few past years. We refer to Balachandran and Chandrasekaran [2], Byszewski [11], [12], Ntouyas [26] and Ntouyas and Tsamatos [24], [25].

Initial value problems for second order semilinear equations with nonlocal conditions, were studied in Ntouyas and Tsamatos [25] and Ntouyas [26].

Recently, the authors in [4] studied the controllability of second order differential inclusions in Banach spaces with nonlocal conditions, in the case where the multivalued map has bounded, closed and convex values, by using the fixed point theorem of Martelli [23]. In this paper, we extend the results of [4] to second order delay integrodifferential inclusions in Banach spaces with nonlocal conditions, when the multivalued F has convex or nonconvex values. In the first case a fixed point theorem due to Martelli is used and in the later a fixed point theorem for contraction multivalued maps, due to Covitz and Nadler [15].

For other recent controllability results for first and second order differential and integrodifferential inclusions in Banach spaces with nonlocal conditions see [5]-[10].

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$C(J,E)$ is the Banach space of continuous functions from J into E normed by

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E.

A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [29]).

$L^1(J,E)$ denotes the Banach space of measurable functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt \quad \text{for all} \quad y \in L^1(J,E).$$

Let $(X, |\cdot|)$ be a Banach space. A multivalued map $G : X \longrightarrow \mathcal{P}(E)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for any bounded set B of X, that is $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$. G is called *upper semicontinuous (u.s.c.)* on X if, for each $x_0 \in X$, the set $G(x_0)$ is a nonempty, closed subset of X, and if, for each open set V of X containing $G(x_0)$, there exists an open neighbourhood U of x_0 such that $G(U) \subseteq V$.

G is said to be *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_0,\ y_n \longrightarrow y_0,\ y_n \in G(x_n)$ imply $y_0 \in G(x_0)$). G has a *fixed point* if there is $x \in X$ such that $x \in G(x)$.

$P(X) = \{Y \in \mathcal{P}(X) : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$, and $P_c(X) = \{Y \in P(X) : Y \text{ convex}\}$. A multivalued map $G : J \longrightarrow P_{cl}(X)$ is said to be *measurable* if for each $x \in X$ the function $Y : J \longrightarrow \mathbb{R}_+$, defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\},$$

is measurable. For more details on multivalued maps we refer to the books of Deimling [16], Gorniewicz [19] and Hu and Papageorgiou [21].

An upper semi-continuous map $G : X \longrightarrow \mathcal{P}(X)$ is said to be *condensing* if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompacteness. For properties of the Kuratowski measure, we refer to Banas and Goebel [3]. We remark that a completely continuous multivalued map is the easiest example of a condensing map.

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if
(i) $C(0) = I$ (I is the identity operator in E),
(ii) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$,
(iii) the map $t \longmapsto C(t)y$ is strongly continuous for each $y \in E$;

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y\,ds, \quad y \in E,\ t \in \mathbb{R}.$$

The infinitesimal generator $A : E \longrightarrow E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2} C(t)y\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the book of Goldstein [18], Heikkila and Lakshmikantham [20], Fattorini [17] and to the papers of Travis and Webb [27], [28].

3 Existence result: The convex case

Assume in this section that $F : J \times E \to \mathcal{P}(E)$ is a bounded, closed and convex valued multivalued map.

We will need the following assumptions:

(H1) A is the infinitesimal generator of a given strongly continuous and bounded cosine family $\{C(t) : t \in J\}$ with $M = \sup\{|C(t)|; t \in J\}$;

(H2) $F : J \times E \longrightarrow P_{b,cl,c}(E); (t, y) \longmapsto F(t, y)$ is measurable with respect to t for each $y \in E$, u.s.c. with respect to y for each $t \in J$ and for each fixed $y \in C(J, E)$ the set

$$S_{F,y} = \left\{g \in L^1(J, E) : \ g(t) \in F(t, y(\sigma(t))) \text{ for a.e. } t \in J\right\}$$

is nonempty;

(H3) there exists a constant L such that

$$|f(y)| \leq L, \quad \text{for each} \ \ y \in C(J, E);$$

(H4) for each $t \in J$, $K(t, s)$ is measurable on $[0, t]$ and

$$K(t) = \text{ess sup}\{|K(t, s)|, \quad 0 \leq s \leq t\},$$

is bounded on J;

(H5) the map $t \longmapsto K_t$ is continuous from J to $L^\infty(J, \mathbb{R})$; here $K_t(s) = K(t, s)$;

(H6) $\sigma : J \to J$ is a continuous function, such that $\sigma(t) \leq t, \forall t \in J$.

(H7) The linear operator $W : L^2(J, U) \to E$, defined by

$$Wu = \int_0^b S(b - s)Bu(s)\, ds,$$

has an invertible operator $\widetilde{W}^{-1}$ which takes values in $L^2(J, U)/kerW$ and there exist positive constants M_1 and M_2 such that $|B| \leq M_1$ and $|W^{-1}| \leq M_2$.

(H8) $\|F(t, y)\| := \sup\{|v| : v \in F(t, y)\} \leq p(t)\psi(|y|)$ for almost all $t \in J$ and all $y \in E$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow (0, \infty)$ is continuous and increasing with

$$Mb \sup_{t \in J} K(t) \int_0^b p(s)ds < \int_c^\infty \frac{du}{\psi(u)};$$

where $c = M(|y_0| + L + Mb|\eta| + M_0)$ and

$$\begin{aligned} M_0 \ &= bM_1M_2\Big[|x_1| + L + M|y_0| + ML + bM|\eta| \\ &+bM \sup_{t \in J} K(t) \int_0^b p(s)\psi(|y(s)|)ds\Big]. \end{aligned}$$

(H9) For each bounded set $D \subset C(J,E)$, and $t \in J$ the set

$$\left\{C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)\int_0^s K(s,u)g(u)duds : g \in S_{F,D}\right\}$$

is relatively compact in E, where $S_{F,D} = \cup\{S_{F,y} : y \in D\}$.

Remark 3.1 *(i) If $dimE < \infty$, then for each $y \in C(J,E)$ the set $S_{F,y}$ is nonempty (see Lasota and Opial [22]).*
(ii) If $dimE = \infty$ and $y \in C(J,E)$ the set $S_{F,y}$ is nonempty if and only if the function $Y : J \longrightarrow \mathbb{R}$ defined by

$$Y(t) := \inf\{|v| : v \in F(t,y)\}$$

belongs to $L^1(J,\mathbb{R})$ (see Hu and Papageorgiou [21]).
(iii) Examples with $W : L^2(J,U) \to E$ such that W^{-1} exists and is bounded are discussed in [13].
(iv) If we assume that $C(t)$, $t > 0$ is completely continuous then (H9) is satisfied.

Definition 3.1 *A function $y \in C(J,E)$ is said to be a mild solution of (1)-(2) on J if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y(\sigma(t)))$ a.e. on J, $y(0) + f(y) = y_0$, and*

$$\begin{aligned} y(t) &= C(t)y_0 - C(t)f(y) + S(t)\eta + \int_0^t S(t-s)(Bu)(s)\,ds \\ &\quad + \int_0^t S(t-s)\int_0^s K(s,\tau)v(\tau)d\tau ds. \end{aligned}$$

Definition 3.2 *The system (1)-(2) is said to be nonlocally controllable on the interval J, if for every $y_0, \eta, x_1 \in E$, there exists a control $u \in L^2(J,U)$, such that the mild solution $y(t)$ of (1)-(2) satisfies $y(b) + f(y) = x_1$.*

The following lemmas are crucial in the proof of our main theorem.

Lemma 3.1 *[22] Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying (H2) and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$, then the operator*

$$\Gamma \circ S_F : C(I,X) \longrightarrow P_{b,cl,c}(C(I,X)),\; y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 3.2 *[23]. Let X be a Banach space and $N : X \longrightarrow P_{b,cl,c}(X)$ an u.s.c. condensing map. If the set*

$$\Omega := \{y \in X : \lambda y \in N(y) \;\; \textit{for some} \;\; \lambda > 1\}$$

is bounded, then N has a fixed point.

Theorem 3.1 *Let $f : C(J,E) \longrightarrow E$ be a continuous function. Assume that hypotheses (H1)-(H9) are satisfied. Then the problem (1)-(2) is nonlocally controllable on J.*

Proof. Using hypothesis (H7) for an arbirtary function $y(\cdot)$ define the control

$$\begin{aligned} u_y(t) &= \widetilde{W}^{-1}\Big[x_1 - f(y) - C(b)y_0 + C(b)f(y) - S(b)\eta \\ &\quad - \int_0^b S(b-s)\int_0^s K(s,\tau)g(\tau)\,d\tau ds\Big](t), \end{aligned}$$

where

$$g \in S_{F,y} = \Big\{g \in L^1(J,E) : g(t) \in F(t, y(\sigma(t))) \ \text{ for a.e. } \ t \in J\Big\}.$$

We shall now show that when using this control, the operator $N : C(J,E) \longrightarrow \mathcal{P}(C(J,E))$ defined by

$$\begin{aligned} N(y) := \Big\{h \in C(J,E) : h(t) &= C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)\,d\tau ds : g \in S_{F,y}\Big\} \end{aligned}$$

has a fixed point. This fixed point is then a solution of the system (1)-(2).

Clearly $x_1 - f(y) \in N(y)(b)$.

We shall show that N satisfies the assumtions of Lemma 3.2. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in C(J,E)$.*

Indeed, if $h_1,\ h_2$ belong to $N(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$\begin{aligned} h_i(t) &= C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)\,ds \\ &\quad + \int_0^t S(t-s)\int_0^s K(s,\tau)g_i(\tau)d\tau ds, \quad i = 1, 2. \end{aligned}$$

Let $0 \le \alpha \le 1$. Then for each $t \in J$ we have

$$\begin{aligned} (\alpha h_1 + (1-\alpha)h_2)(t) &= C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s K(s,\tau)[\alpha g_1(\tau) + (1-\alpha)g_2(\tau)]d\tau ds. \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values) then

$$\alpha h_1 + (1-\alpha)h_2 \in N(y).$$

Step 2: *N is bounded on bounded sets of $C(J,E)$.*

Indeed, it is enough to show that there exists a positive constant l such that for each $h \in N(y), y \in B_r = \{y \in C(J,E) : \|y\|_\infty \leq r\}$ one has $\|h\|_\infty \leq l$. If $h \in N(y)$, then there exists $g \in S_{F,y}$ such that

$$\begin{aligned} h(t) &= C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)\,ds \\ &\quad + \int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)d\tau ds, \quad t \in J. \end{aligned}$$

By (H3)-(H7) we have for each $t \in J$ that

$$\begin{aligned} |h(t)| &\leq |C(t)||y_0| + |C(t)||f(y)| + |S(t)||\eta| + \left\|\int_0^t S(t-s)(Bu_y)(s)\,ds\right\| \\ &\quad + \left\|\int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)d\tau ds\right\| \\ &\leq M|y_0| + ML + Mb|\eta| \\ &\quad + bMM_1M_2[|x_1| + L + M|y_0| + ML + bM|\eta| \\ &\quad + Mb\sup_{t\in J} K(t)\|p\|_{L^1}\sup_{t\in J}\psi(|y(t)|)] \\ &\quad + M\int_0^t\int_0^s |K(s,u)|p(u)\psi(|y(\sigma(u))|)duds \\ &\leq M|y_0| + ML + b|\eta| \\ &\quad + bMM_1M_2[|x_1| + L + M|y_0| + ML + bM|\eta| \\ &\quad + Mb\sup_{t\in J} K(t)\|p\|_{L^1}\sup_{t\in J}\psi(|y(t)|)] \\ &\quad + Mb\sup_{t\in J} K(t)|p|_{L^1}\sup_{t\in J}\psi(|y(t)|). \end{aligned}$$

Then for each $h \in N(y)$ we have

$$\begin{aligned} \|h\|_\infty &\leq M|y_0| + ML + b|\eta| + MM_0 + Mb\sup_{t\in J} K(t)\|p\|_{L^1}\sup_{t\in J}\psi(|y(t)|)] \\ &\quad + Mb\sup_{t\in J} K(t)|p|_{L^1}\sup_{t\in J}\psi(|y(t)|) := l. \end{aligned}$$

Step 3: *N sends bounded sets of $C(J,E)$ into equicontinuous sets.*

Let $t_1, t_2 \in J, t_1 < t_2$ and B_r be a bounded set of $C(J,E)$. For each $y \in B_r$ and $h \in N(y)$, there exists $g \in S_{F,y}$ such that

$$h(t) = C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)ds$$

$$+\int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)d\tau ds, \quad t\in J.$$

Thus

$$\begin{aligned}
|h(t_2)-h(t_1)| &\leq |(C(t_2)-C(t_1))y_0| + L|C(t_2)-C(t_1)| + |S(t_2)-S(t_1)||\eta| \\
&+\Big\|\int_0^{t_2}[S(t_2-s)-S(t_1-s)]BW^{-1}\big[x_1-f(y)-C(b)y_0 \\
&+C(b)f(y)-S(b)\eta-\int_0^b S(b-s)\int_0^s K(s,\tau)g(\tau)d\tau ds\big](\eta)d\eta\Big\| \\
&+\Big\|\int_{t_1}^{t_2} S(t_1-s)BW^{-1}\big[x_1-f(y)-C(b)y_0+C(b)f(y)-S(b)\eta \\
&+\int_0^b S(b-s)\int_0^s K(s,\tau)g(\tau)d\tau ds\big](\eta)d\eta\Big\| \\
&+\Big\|\int_0^{t_2}[S(t_2-s)-S(t_1-s)]\int_0^s K(s,\tau)g(\tau)d\tau ds\Big\| \\
&+\Big\|\int_{t_1}^{t_2} S(t_1-s)\int_0^s K(s,\tau)g(\tau)d\tau ds\Big\| \\
&\leq |C(t_2)-C(t_1)||y_0| + L|C(t_2)-C(t_1)| + |S(t_2)-S(t_1)||\eta| \\
&+\int_0^{t_2}|S(t_2-s)-S(t_1-s)|\,M_1M_2\big[|x_1|+L+M|y_0| \\
&+ML+bM|\eta|+Mb\sup_{t\in J}K(t)\int_0^b p(s)\psi(|y(s)|)ds\big](\eta)d\eta \\
&+\int_{t_1}^{t_2}|S(t_1-s)|M_1M_2\big[|x_1|+L+M|y_0|+ML+bM|\eta| \\
&+Mb\sup_{t\in J}K(t)\int_0^b p(s)\psi(|y(s)|)ds\big](\eta)d\eta \\
&+\sup_{t\in J}K(t)\Big\|\int_0^{t_2}[S(t_2-s)-S(t_1-s)]\int_0^s g(\tau)d\tau ds\Big\| \\
&+M\sup_{t\in J}K(t)(t_2-t_1)\int_0^b \|g(s)\|ds.
\end{aligned}$$

As $t_2 \longrightarrow t_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Step 2, Step 3 and (H9), together with the Arzela-Ascoli theorem, we can conclude that N is completely continuous, and therefore, a condensing multivalued map.

Step 4: *N has a closed graph.*

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) &= C(t)y_0 - C(t)f(y_n) + S(t)\eta + \int_0^t S(t-s)(Bu_{y_n})(s)ds \\ &\quad + \int_0^t S(t-s)\int_0^s K(s,\tau)g_n(\tau)d\tau ds, \quad t \in J, \end{aligned}$$

where

$$\begin{aligned} u_{y_n}(t) &= \widetilde{W}^{-1}\Big[\eta - f(y_n) - C(b)y_0 + C(b)f(y_n) - S(b)\eta \\ &\quad - \int_0^b S(b-s)\int_0^s K(s,\tau)g_n(\tau)\, d\tau ds\Big](t). \end{aligned}$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that

$$\begin{aligned} h_*(t) &= C(t)y_0 - C(t)f(y_*) + S(t)\eta + \int_0^t S(t-s)(Bu_{y_*})(s)ds \\ &\quad + \int_0^t S(t-s)\int_0^s K(s,\tau)g_*(\tau)d\tau ds, \quad t \in J, \end{aligned}$$

where

$$\begin{aligned} u_{y_*}(t) &= \widetilde{W}^{-1}\Big[\eta - f(y_*) - C(b)y_0 + C(b)f(y_*) - S(b)\eta \\ &\quad - \int_0^b S(b-s)\int_0^s K(s,\tau)g_*(\tau)\, d\tau ds\Big](t). \end{aligned}$$

Set

$$\overline{u}_y(t) = \widetilde{W}^{-1}\Big[\eta - f(y) - C(b)y_0 + C(b)f(y) - S(b)\eta\Big].$$

Since f, W^{-1} are continuous, then $\overline{u}_{y_n}(t) \longrightarrow \overline{u}_{y_*}(t)$ for $t \in J$.
Clearly we have that

$$\begin{aligned} &\Big\|\Big(h_n - C(t)y_0 + C(t)f(y_n) - S(t)\eta - \int_0^t S(t-s)(B\overline{u}_{y_n})(s)ds\Big) \\ &- \Big(h_* - C(t)y_0 + C(t)f(y_*) - S(t)\eta - \int_0^t S(t-s)(B\overline{u}_{y_*})(s)ds\Big)\Big\|_\infty \to 0, \end{aligned}$$

as $n \longrightarrow \infty$.

Consider the operator

$$\Gamma : L^1(J,E) \longrightarrow C(J,E)$$

$$\begin{aligned} g \longmapsto \Gamma(g)(t) &= \int_0^t S(t-s)\big[BW^{-1}\Big(\int_0^b S(b-s)\int_0^s K(s,\tau)g(\tau)d\tau ds\Big)(s)\big]ds \\ &+ \int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)d\tau ds. \end{aligned}$$

Clearly, Γ is linear and continuous. Indeed one has

$$\|\Gamma g\|_\infty \leq b^2 M \sup_{t\in J} K(t)(bMM_1M_2+1)\|g\|_{L^1}.$$

¿From Lemma 3.2, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have that

$$h_n(t) - C(t)y_0 + C(t)f(y_n) - S(t)\eta - \int_0^t S(t-s)(B\overline{u}_{y_n})(s)ds \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows from Lemma 3.2 that

$$\begin{aligned} & h_*(t) - C(t)y_0 + C(t)f(y_*) - S(t)\eta - \int_0^t S(t-s)(B\overline{u}_{y_*})(s)ds \\ = & \int_0^t S(t-s)\big[BW^{-1}\Big(\int_0^b S(b-s)\int_0^s K(s,\tau)g_*(\tau)d\tau ds\Big)(s)\big]ds \\ + & \int_0^t S(t-s)\int_0^s K(s,\tau)g_*(\tau)d\tau ds \end{aligned}$$

for some $g_* \in S_{F,y_*}$.

Step 5: *The set*

$$\Omega := \{y \in C(J,E) : \lambda y \in N(y), \quad \textit{for some} \quad \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$\begin{aligned} y(t) &= \lambda^{-1}C(t)y_0 - \lambda^{-1}C(t)f(y) + \lambda^{-1}S(t)\eta \\ &+\lambda^{-1}\int_0^t S(t-s)BW^{-1}\Big[x_1 - f(y) - C(b)y_0 + C(b)f(y) + S(t)\eta \\ &- \int_0^b S(b-s)\int_0^s K(s,\tau)g(\tau)\,d\tau ds\Big](\eta)d\eta \\ &+\lambda^{-1}\int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)d\tau ds, \ t \in J. \end{aligned}$$

This implies by (H3)-(H8) that for each $t \in J$ we have

$$|y(t)| \;\leq\; M|y_0| + ML + bM|\eta|$$

$$
\begin{aligned}
&+bMM_1M_2\Big[|x_1| + L + M|y_0| + ML + bM|\eta| \\
&+bM\sup_{t\in J} K(t)\int_0^b p(s)\psi(|y(\sigma(s))|)ds\Big] \\
&+M\left|\int_0^t\int_0^s K(s,\tau)g(\tau)d\tau ds\right| \\
\leq\ & M|y_0| + ML + bM|\eta| \\
&+bMM_1M_2\Big[|x_1| + L + M|y_0| + ML + bM|\eta| \\
&+bM\sup_{t\in J} K(t)\int_0^b p(s)\psi(|y(\sigma(s))|)ds\Big] \\
&+Mb\sup_{t\in J} K(t)\int_0^t p(s)\psi(|y(\sigma(s))|)ds.
\end{aligned}
$$

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$v(0) = M|y_0| + ML + MM_0, \ \ |y(t)| \leq v(t), \ t \in J,$$

and

$$v'(t) = Mb\sup_{t\in J} K(t)p(t)\psi(|y(\sigma(t))|), \ t \in J.$$

Using the nondecreasing character of ψ and the fact that $\sigma(t) \leq t, \forall t \in J$ we get

$$v'(t) \leq Mb\sup_{t\in J} K(t)p(t)\psi(v(t)), \ t \in J.$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq Mb\sup_{t\in J} K(t)\int_0^b p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

This inequality implies that there exists a constant d such that $v(t) \leq d, \ t \in J$, and hence $\|y\|_\infty \leq d$ where d depends only on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J, E)$. As a consequence of Lemma 3.2 we deduce that N has a fixed point and thus the system (1)-(2) is nonlocally controllable on J. ■

4 Existence Result: The nonconvex case

In this section we consider the problems (1)-(2), with a nonconvex valued right hand side.

Let (X,d) be the metric space induced from the normed space $(X,|\cdot|)$.
Consider $H_d: P(X)\times P(X)\longrightarrow \mathbb{R}_+\cup\{\infty\}$, given by

$$H_d(A,B)=\max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b)=\inf_{a\in A} d(a,b),\ \ d(a,B)=\inf_{b\in B} d(a,b)$.
Then $(P_{b,cl}(X),H_d)$ is a metric space and $(P_{cl}(X),H_d)$ is a generalized metric space.

Definition 4.1 *A multivalued operator $N: X\to P_{cl}(X)$ is called*

a) γ-Lipschitz if and only if there exists $\gamma>0$ such that

$$H_d(N(x),N(y))\le \gamma d(x,y), \quad \textit{for each } x,\ y\in X,$$

b) contraction if and only if it is γ-Lipschitz with $\gamma<1$.

c) N has a fixed point *if there is $x\in X$ such that $x\in N(x)$. The fixed point set of the multivalued operator N will be denoted by $FixN$.*

Our considerations are based on the following fixed point theorem for contraction multivalued operators given by Covitz and Nadler in 1970 [15] (see also Deimling, [16] Theorem 11.1).

Lemma 4.1 *Let (X,d) be a complete metric space. If $N: X\to P_{cl}(X)$ is a contraction, then $FixN\ne\emptyset$.*

We will need the following assumptions:

(A1) $F:[0,b]\times E\longrightarrow P_{cl}(E)$ is integrably bounded and has the property that $F(\cdot,y):[0,b]\to P_{cl}(E)$ is measurable for each $y\in E$.

(A2) $H_d(F(t,y),F(t,\overline{y}))\le l(t)\|y-\overline{y}\|$, for almost each $t\in[0,b]$ and $y,\overline{y}\in E$, where $l\in L^1([0,b],\mathbb{R})$.

(A3) $\|f(y)-f(\overline{y})\|\le c\|y-\overline{y}\|$, for each $t\in[0,b]$ and $y,\overline{y}\in C([0,b],E)$, where c is a nonnegative constant.

Now we are able to state and prove our main result for this section.

Theorem 4.1 *Assume that hypotheses (H1), (H4)-(H7) and (A1)-(A3) are satisfied. Then the problem (1)-(2) is nonlocally controllable on J, provided*

$$Mc+MM_1M_2bc+M^2M_1M_2bc+\frac{M^2M_1M_2b^2\sup_{t\in J}K(t)}{\tau}+\frac{M}{\tau}<1.$$

Proof. Using hypothesis (H7) for an arbirtary function $y(\cdot)$ define the control

$$\begin{aligned} u_y(t) &= \widetilde{W}^{-1}\Big[x_1 - f(y) - C(b)y_0 + C(b)f(y) - S(b)\eta \\ &\quad - \int_0^b S(b-s)\int_0^s K(s,\tau)g(\tau)\,d\tau ds\Big](t), \end{aligned}$$

where $g \in S_{F,y}$.

Remark 4.1 *For each $y \in C([0,b],E)$, the set $S_{F,y}$ is nonempty, since by (A1), F has a measurable selection (see [14], Theorem III.6).*

We shall now show that, when using this control, the operator $N : C(J,E) \longrightarrow \mathcal{P}(C(J,E))$ defined by

$$\begin{aligned} N(y) := \Big\{h \in C(J,E) : h(t) &= C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s K(s,\tau)g(\tau)d\tau ds : g \in S_{F,y}\Big\} \end{aligned}$$

has a fixed point. This fixed point is then a solution of the system (1)-(2).

Clearly $x_1 - f(y) \in N(y)(b)$.

We shall show that N satisfies the assumptions of Lemma 4.1. The proof will be given in two steps.

Step 1: *$N(y) \in P_{cl}(C([0,b],E))$ for each $y \in C([0,b],E)$.*

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \longrightarrow \tilde{y}$ in $C([0,b],E)$. Then $\tilde{y} \in C([0,b],E)$ and

$$\begin{aligned} y_n(t) &\in C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s K(s,\tau)F(\tau, y(\sigma(\tau)))d\tau ds, \ t \in J. \end{aligned}$$

Using (A1) one can easily show by standard argument that

$$\int_0^t S(t-s)\int_0^s K(s,\tau)F(\tau, y(\sigma(\tau)))d\tau ds$$

is closed for each $t \in [0,b]$. Then

$$\begin{aligned} y_n(t) \longrightarrow \tilde{y}(t) &\in C(t)(y_0 - f(y)) + S(t)\eta + \int_0^t S(t-s)(Bu_y)(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s K(s,\tau)F(\tau, y(\sigma(\tau)))d\tau ds, \ t \in J. \end{aligned}$$

So $\tilde{y} \in N(y)$.

Step 2: *$H_d(N(\eta), N(y_2)) \leq \gamma\|y_1 - y_2\|$ for each $y_1, y_2 \in C([0,b],E)$ (where $\gamma < 1$).*

Let $y_1, y_2 \in C([0,b], E)$ and $h_1 \in N(y_1)$. Then there exists $g_1(t) \in F(t, y_1(\sigma(t)))$ such that

$$\begin{aligned} h_1(t) &= C(t)(y_0 - f(y_1)) + S(t)\eta + \int_0^t S(t-s)(Bu_{y_1})(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s g_1(\tau)d\tau ds, \ t \in J. \end{aligned}$$

From (H3) it follows that

$$\begin{aligned} H_d(F(t, y_1(\sigma(t))), F(t, y_2(\sigma(t)))) &\le l(t)\|y_1(\sigma(t)) - y_2(\sigma(t))\| \\ &\le l(t)\|y_1(t) - y_2(t)\|, \ t \in J. \end{aligned}$$

Hence there is $w \in F(t, y_2(\sigma(t)))$ such that

$$\|g_1(t) - w\| \le l(t)\|y_1(t) - y_2(t)\|, \quad t \in J.$$

Consider $U : [0,b] \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : \|g_1(t) - w\| \le l(t)\|y_1(t) - y_2(t)\|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, y_2(\sigma(t)))$ is measurable (see Proposition III.4 in [14]) there exists $g_2(t)$ a measurable selection for V. So, $g_2(t) \in F(t, y_2(\sigma(t)))$ and

$$\|g_1(t) - g_2(t)\| \le l(t)\|y_1(t) - y_2(t)\|, \quad \text{for each } \ t \in J.$$

Let us define for each $t \in J$

$$\begin{aligned} h_2(t) &= C(t)(y_0 - f(y_2)) + S(t)\eta + \int_0^t S(t-s)(Bu_{y_2})(s)\,ds \\ &+ \int_0^t S(t-s)\int_0^s g_2(\tau)d\tau ds. \end{aligned}$$

Then we have

$$\begin{aligned} \|h_1(t) - h_2(t)\| &\le M\|f(y_1) - f(y_2)\| + MM_1\int_0^t \|u_{y_1}(s) - u_{y_2}(s)\|ds \\ &+ Mb\int_0^t \|g_1(s) - g_2(s)\|\,ds \\ &\le Mc\|y_1 - y_2\| + MM_1M_2\int_0^t [\|f(y_1) - f(y_2)\| \\ &+ M\|f(y_1) - f(y_2)\| + b^2 \sup_{t \in J} K(t)M\|g_1(s) - g_2(s)\|]ds \\ &+ Mb\int_0^t \|g_1(s) - g_2(s)\|ds \end{aligned}$$

$$\begin{aligned}
&\le Mc\|y_1 - y_2\| + MM_1M_2[bc\|y_1 - y_2\| + bMc\|y_1 - y_2\| \\
&\quad + b^2 \sup_{t\in J} M \int_0^t l(s)\|y_1(s) - y_2(s)\|ds] \\
&\quad + M \int_0^t l(s)\|y_1(s) - y_2(s)\|ds \\
&= Mce^{\tau L(t)}\|y_1 - y_2\|_B + MM_1M_2bce^{\tau L(t)}\|y_1 - y_2\|_B \\
&\quad + M^2M_1M_2bce^{\tau L(t)}\|y_1 - y_2\|_B \\
&\quad + M^2M_1M_2b^2 \sup_{t\in J} K(t) \int_0^t l(s)e^{-\tau L(s)}e^{\tau L(s)}\|y_1(s) - y_2(s)\|ds \\
&\quad + M \int_0^t l(s)e^{-\tau L(s)}e^{\tau L(s)}\|y_1(s) - y_2(s)\|ds \\
&\le Mce^{\tau L(t)}\|y_1 - y_2\|_B + MM_1M_2bce^{\tau L(t)}\|y_1 - y_2\|_B \\
&\quad + M^2M_1M_2bce^{\tau L(t)}\|y_1 - y_2\|_B \\
&\quad + M^2M_1M_2b^2 \sup_{t\in J} K(t)\|y_1 - y_2\|_B \frac{1}{\tau}e^{\tau L(t)} \\
&\quad + M\|y_1 - y_2\|_B \frac{1}{\tau}e^{\tau L(t)},
\end{aligned}$$

where $L(t) = \int_0^t l(s)ds$, and $\|\cdot\|_B$ is the Bielecki-type norm on $C([0,b],E)$ defined by

$$\|y\|_B = \max_{t\in[0,b]}\{\|y(t)\|e^{-\tau L(t)}\}.$$

Then

$$\|h_1-h_2\|_B \le \left[Mc + MM_1M_2bc + M^2M_1M_2bc + \frac{M^2M_1M_2b^2 \sup_{t\in J} K(t)}{\tau} + \frac{M}{\tau}\right]\|y_1-y_2\|_B.$$

By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H_d(N(y_1), N(y_2)) \le$$

$$\left[Mc + MM_1M_2bc + M^2M_1M_2bc + \frac{M^2M_1M_2b^2 \sup_{t\in J} K(t)}{\tau} + \frac{M}{\tau}\right]\|y_1 - y_2\|_B.$$

Then N is a contraction and thus, by Lemma 4.1, it has a fixed point y, which is a mild solution to (1)-(2). ■

5 An Example

Consider the following partial integrodifferential equation of the form

$$\begin{aligned} z_{tt}(t,y) - z_{yy}(t,y) &= \int_0^t K(t,s)q(s,z(s,y(s-\tau)))\,ds \\ &\quad +Bu(t), \ \ 0 \leq y \leq \pi, \ t \in J \end{aligned} \tag{3}$$

$$\begin{aligned} z(t,0) &= z(t,\pi) = 0, \\ z(0,y) + z(1,y) &= z_0(y), \\ z_t(0,y) &= z_1(y) \end{aligned} \tag{4}$$

where $q : J \times E \to E$, is continuous.

Let $E = L^2[0,\pi]$ and define $A : E \to E$ by $Aw = w''$ with domain

$$D(A) = \{w \in E, w, w' \ \text{ are absolutely continuous, } w'' \in E, w(0) = w(\pi) = 0\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} n^2(w, w_n)w_n, \ \ w \in D(A)$$

where $w_n(s) = \sqrt{\frac{2}{\pi}}\, sin\, ns$, $n = 1, 2, \ldots$ is the orthogonal set of eigenvectors in A. It is easily shown that A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in \mathbb{R}$ in E given by

$$C(t)w = \sum_{n=1}^{\infty} cosnt(w, w_n)w_n, \quad w \in E$$

and that the associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{1}{n} sinnt(w, w_n)w_n, \quad w \in E.$$

Assume that the operator $B : U \to Y, U \subset J$, is a bounded linear operator and the operator

$$Wu = \int_0^b S(b-s)Bu(s)ds$$

has a bounded invertible operator $\widetilde{W}^{-1}$ which takes values in $L^2(J,U)\backslash kerW$.

Assume that there exists an integrable function $p : J \to [0, \infty)$ such that

$$|q(t, w(t))| \leq p(t)\psi(|w|)$$

where $\psi : [0,\infty) \to (0,\infty)$ is continuous and nondecreasing with

$$Mb \sup_{t\in J} K(t) \int_0^b p(t)dt < \int_c^{\infty} \frac{ds}{\psi(s)}$$

where c is a constant.

Then the problem (1)-(2) is an abstract formulation of (3)-(4). Since all the conditions of Theorem 4.1 are satisfied, the problem (3)-(4) is nonlocally controllable on J.

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